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On the Deformation of Thin Elastic Plates and Shells.

By A. B. Basset, M. A., F. R. S.

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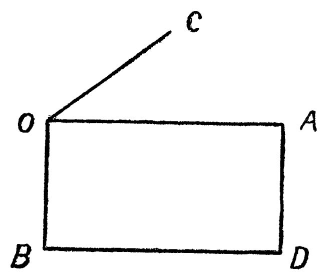
1. The mathematical theory of the deformation of a thin elastic plate or shell involves difficulties of a formidable nature. This is partly owing to the fact that an approximate solution, which is correct as far as terms involving the cube of the thickness of the plate or shell, cannot be obtained without having recourse to an extremely lengthy and troublesome process which requires the Calculus of Variations, and partly because the older writers upon this subject based their investigations upon hypotheses which were in most cases inadequate and erroneous. When a mathematician of standing and reputation gravely propounds a hypothesis which turns out to be incorrect, or condemns as unsound some method which is perfectly legitimate, and in addition obtains by means of his erroneous hypothesis many results which are substantially correct, the mischief done to science can hardly be exaggerated; for subsequent investigators not only are led astray from the path which must be followed in order to obtain a satisfactory theory, but are hampered by the difficulty of convincing mankind of the errors of their predecessors.

The development of the theory of thin elastic plates and shells has been retarded by two errors for which Clebsch and Saint-Venant are mainly responsible. The first error is that the three stresses R, S, T are accurately zero throughout the substance of the plate or shell; the second error is that it is not permissible to expand the various quantities involved, in a series of ascending powers of the distance of a point from the middle surface. In England the first error may, I think, now be regarded as exploded; but the second one has not yet been completely driven from the field, for a perusal of Mr. Love's recent Treatise on Elasticity shows that he entertains some bias against the method of expansion, though on what grounds I am at a loss to conceive.

Owing to the unsatisfactory manner in which Mr. Love has dealt with the theories of thin plates, shells and wires in his book, I propose in the present paper to give as concise an account of the first two theories as the nature

of the case will admit. I shall commence by expounding the fundamental principles of these theories; I shall then proceed to develop them in a form suitable for mathematical calculation; I shall endeavor to avoid all unnecessary mathematical complications; and I shall omit or pass over very lightly those parts which are not of much physical interest, or which require lengthy and difficult analysis for their elucidation.

2. We shall first consider the resultant stresses which act upon any element of a thin shell.



Let $OADB$ be a small curvilinear rectangle described on the middle surface of a thin shell, the sides of which are lines of curvature; and let us consider a small element of the shell bounded by the external surfaces, and the four planes passing through the sides of this rectangle which are perpendicular to the middle surface.

The resultant stresses per unit of length which act upon the element, and which are due to the action of contiguous portions of the shell, are completely specified by the following ten quantities, viz. across the section AD:

$T_1 =$ a tension across AD parallel to OA,
$M_2 =$ a tangential shearing stress along AD,
$N_2 =$ a normal shearing stress parallel to OC,
$G_2 =$ a flexural couple from C to A whose axis is parallel to AD,
$H_1 =$ a torsional couple from B to C whose axis is parallel to OA.

Similarly the resultant stresses per unit of length which act across the section BD are:

$T_2 =$ a tension across BD parallel to OB,
$M_1 =$ a tangential shearing stress along BD,
$N_1 =$ a normal shearing stress parallel to OC,
$G_1 =$ a flexural couple from B to C whose axis is parallel to BD,
$H_2 =$ a torsional couple from C to A whose axis is parallel to OB.

By resolving all the forces and stresses which act upon the element parallel to OA, OB and OC, and by taking moments about these lines, we obtain the six equations of motion of the element; but these equations will not enable us to solve any statical or dynamical problems, since they furnish only six equations connecting ten unknown quantities. In order to complete the solution we require the values of the stresses and the equations of motion in terms of the displacements of a point on the middle surface and their space variations in terms of the coordinates of that point.

The internal stresses at any point of the substance of the shell are completely specified by the six quantities P, Q, R, S, T, U; and at this stage of the investigation it will be well to state that the notation of Thomson and Tait's Natural Philosophy will be employed for internal stresses and elastic constants, but that the three extensional strains parallel to OA, OB, OC will be denoted by σ_1, σ_2, σ_3, and the three shearing strains about these lines by $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$.

3. In most cases of practical importance (although there are a few exceptions), the surfaces of the shell are free from stress, and when this is the case, the three stresses R, S and T must vanish at both surfaces. Now Clebsch and several of the older writers fell into the mistake of supposing that these stresses are *zero throughout the substance of the shell*, and by the aid of this erroneous hypothesis theories were constructed which nevertheless furnished numerous results which were substantially correct. The reason of this was as follows: The theory of thin plates and shells is an approximate one, in which the thickness is supposed to be a small quantity in comparison with the least radius of curvature of the middle surface, and it is sufficient to carry the approximation no farther than powers involving the cube of the thickness. If $2h$ is the thickness and h' the distance of any point of the shell from the middle surface, it will appear as we proceed that if the lowest terms in R, S, T were quadratic functions of h and h', the retention of these stresses would give rise to terms of a higher order than h^3, and consequently an approximate solution could be obtained to that order by *treating* R, S, T as zero. The first question therefore is, have we any evidence which would justify us in concluding that this supposition is true?

4. The periods of the vibrations of a plane plate of infinite extent have been worked out by Lord Rayleigh* by means of the general equations of an

* Proc. Lond. Math. Soc., Vol. XX, p. 225.

elastic solid. The vibrations consist of two distinct types, viz. flexural vibrations unaccompanied by extension of the middle surface in which the displacement is perpendicular to that surface, and extensional vibrations in which the displacements are parallel to the middle surface. If Lord Rayleigh's expressions for these two kinds of vibrations be expanded in powers of the thickness, it will be found that if $2\pi/p_1$, $2\pi/p_2$ be the periods,

$$p_1^2 = \frac{4mnh^2f^4}{3\rho(m+n)} + \ldots \text{ higher powers of } h^2, \tag{1}$$

where n is the rigidity, $m = k + \frac{1}{3}n$, where k is the resistance to compression, ρ the density and $2\pi/f$ the wave-length of the disturbance. The value of p_2 is

$$p_2^2 = \frac{4mnf^2}{\rho(m+n)}\left\{1 - \frac{(m-n)^2f^2h^2}{3(m+n)^2} + \ldots\right\} \tag{2}$$

Now if we attempt to obtain the same results by assuming that the three stresses R, S, T are quadratic functions of h and h', it will be found that these quantities give rise to terms in the expression for the energy which are of a higher order than h^3 and may therefore be neglected, and we shall finally obtain the same expressions for p_1 and p_2 as are given above.*

It can also be proved by means of Lord Rayleigh's paper, that in these two particular cases R, S and T are quadratic functions of h and h'.

In order to test the truth of the hypothesis in the case of a curved shell, I worked† out the period of the radial vibrations of a complete cylindrical shell to a second approximation, so as to obtain the term in h^2, by means of the general equations of elasticity and also by means of the theory of thin plates, and I found that the results obtained by these two methods agreed. The value of p in this case is

$$p^2 = \frac{4mn}{\rho a^2(m+n)}\left\{1 + \frac{h^2}{3a^2}\left[1 + \frac{4(m-n)}{m+n}\right]\right\} \tag{3}$$

where a is the radius of the middle surface.

We notice in passing that in the case of a cylindrical shell the pitch of the purely extensional vibrations rises as the thickness increases, whereas the contrary is the case when the surface is an infinite plane plate. Also the frequency of the flexural vibrations of a plane plate is proportional to the thickness, whereas the frequency of the extensional vibrations contains a term independent

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*See Proc. Lond. Math. Soc., Vol. XXI, p. 51. † *Ibid.* p. 53.

of the thickness; the notes yielded by extensional vibrations are therefore of much higher pitch than those yielded by non-extensional ones.

5. In consequence of the direct evidence furnished by the above-mentioned results, I felt justified, in 1889, in laying down the following fundamental hypothesis as the basis of the true theory of thin plates and shells, viz.:

When the surfaces of a thin plate or shell are not subjected to any surface forces, such as external pressures or tangential stresses, the three stresses R, S, T, so far as they depend upon h and h', are of the form

$$u_2 + u_3 + \ldots . u_n + \ldots . ,$$

where u_n is a homogeneous n^{tic} function of h and h'.

On applying this hypothesis to obtain a theory of plane plates and cylindrical and spherical shells,* I found that expressions for the kinetic and potential energies could be obtained which are correct as far as h^3, and that in deducing these expressions a knowledge of R, S and T was not required, but that they might be treated as zero, inasmuch as they would, if retained, give rise to terms involving powers of h higher than h^3. I was also able to test the correctness of the fundamental hypothesis and also of the work in the following manner:—

As soon as the correct expressions for the potential and kinetic energies have been obtained in terms of the displacements, the equations of motion and the values of the ten sectional stresses can be deduced in terms of these quantities by means of the Principle of Virtual Work.

Now we have already pointed out that the six equations of motion can be written down in terms of the ten sectional stresses, and if we substitute in these equations the values of these stresses in terms of the displacements which we have obtained by means of the Principle of Virtual Work, three of them ought to be identically satisfied, whilst the remaining three ought to be identical with the three equations of motion which we have obtained by means of that principle. This is the first test, and it is satisfied. The second test is that all the sectional stresses except T_1 and T_2 can be calculated by a direct method, and the values so obtained ought to agree with those furnished by the Principle of Virtual Work. This test is also satisfied.

6. Let P' be any point of the shell, P its projection on the middle surface; then all the quantities with which we are concerned are functions of the position

* Phil. Trans., 1890, p. 433.

of P', and are therefore functions of (r, z, ϕ) or (r, θ, ϕ) according as the shell is cylindrical or spherical. If, therefore, $\mathfrak{Q}'$ be the value of any such quantity at P', and $\mathfrak{Q}$ its value at P, it follows that

$$\begin{aligned}\mathfrak{Q}' &= F(r) = F(a + h') \\ &= \mathfrak{Q} + h'\left(\frac{d\mathfrak{Q}}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\mathfrak{Q}}{dr^2}\right) + \ldots\end{aligned}$$

by Taylor's theorem, where the brackets are employed, as will be done throughout this paper, to denote the values of the differential coefficients at the middle surface.

I have already alluded to the fact that objections have been raised by Saint-Venant, and endorsed by Mr. Love,* against the method of expansion. These objections appear to me to be so utterly without foundation that it would be scarcely worth while to labor the point were it not that Mr. Love, throughout his recent treatise, has shown a persistent disinclination to employ this method, although he does not venture to repeat his former arguments. In consequence of this he has been compelled to resort to lengthy and unnecessarily complicated methods which I fear will greatly retard the further development of the subject. It has been known for many years past that when a function and its differential coefficients are finite and continuous between given limits of the variable, the function can always be expanded between these limits by Taylor's theorem, and that we can stop at any term of the series that we please and express the remainder in a finite form. Now although it is quite true that functions may be imagined which, by reason of their becoming infinite or discontinuous between the limits, would render the application of Taylor's theorem inadmissible, yet the occurrence of such functions is precluded by physical considerations, because they would involve a rupture of the material. We all know that bells sometimes crack and wires break; but such catastrophes are not contemplated, and would require a special theory to account for them. The sooner these objections are consigned to oblivion the better it will be for the theory of Elasticity.

We are now in a position to consider the mathematical part of the theory.

Theory of Plane Plates.

7. We shall commence by obtaining the equations of motion in terms of the sectional stresses.

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* Phil. Trans., 1888, p. 491.

Let u', v', w' be the displacements parallel to OA, OB, OC of any point P' of the plate; X, Y, Z the components of the bodily forces per unit of mass. Then taking moments about OC, it follows at once that $M_1 = M_2$; also resolving parallel to OA, OB, OC and taking moments about OA, OB, we obtain the following five equations, viz.:

$$\left.\begin{aligned}
\frac{dT_1}{dx} + \frac{dM}{dy} &= \rho\int_{-h}^{h}\ddot{u}'dh' - 2\rho hX, \\
\frac{dT_2}{dy} + \frac{dM}{dx} &= \rho\int_{-h}^{h}\ddot{v}'dh' - 2\rho hY, \\
\frac{dN_2}{dx} + \frac{dN_1}{dy} &= \rho\int_{-h}^{h}\ddot{w}'dh' - 2\rho hZ, \\
\frac{dG_1}{dy} + \frac{dH_1}{dx} + N_1 &= -\rho\int_{-h}^{h}\ddot{v}'h'dh', \\
\frac{dG_2}{dx} + \frac{dH_2}{dy} - N_2 &= \rho\int_{-h}^{h}\ddot{u}'h'dh'.
\end{aligned}\right\} \tag{1}$$

8. We shall now proceed to calculate the four couples in terms of the displacements of a point on the middle surface.

Let P be the projection of P' on the middle surface, and let the values of the different quantities at P' and P be distinguished by accented and unaccented letters. The equations determining the six strains in terms of the displacements are

$$\left.\begin{aligned}
\sigma_1' &= \frac{du'}{dx}, & \sigma_2' &= \frac{dv'}{dy}, & \sigma_3' &= \frac{dw'}{dz}, \\
\bar{\omega}_1' &= \frac{dv'}{dz} + \frac{dw'}{dy}, & \bar{\omega}_2' &= \frac{dw'}{dx} + \frac{du'}{dz}, & \bar{\omega}_3' &= \frac{du'}{dy} + \frac{dv'}{dx}.
\end{aligned}\right\} \tag{2}$$

The value of R', when expanded in powers of h', may be written

$$R' = A_0 + A_1h' + \tfrac{1}{2}A_2h'^2 + \ldots\ldots, \tag{3}$$

where, according to the fundamental hypothesis, A_0 and A_1 do not contain any lower powers of h than h^2 and h respectively. But

$$\begin{aligned}
R' &= (m+n)\,\sigma_3' + (m-n)(\sigma_1' + \sigma_2') \\
&= (m+n)\,\sigma_3 + (m-n)(\sigma_1 + \sigma_2) \\
&\quad + \left\{(m+n)\left(\frac{d\sigma_3}{dz}\right) + (m-n)(\lambda + \mu)\right\}h' + \ldots\ldots,
\end{aligned} \tag{4}$$

where λ and μ denote the values of $d\sigma_1/dz$ and $d\sigma_2/dz$ when $z=0$. Accordingly, by (3) and (4),

$$\left.\begin{aligned}(m+n)\,\sigma_3 \quad &+ (m-n)(\sigma_1+\sigma_2) = A_0,\\ (m+n)\left(\frac{d\sigma_3}{dz}\right) &+ (m-n)(\lambda+\mu) = A_1,\end{aligned}\right\} \tag{5}$$

The value of G_1 is

$$G_1 = -\int_{-h}^{h} Q'h'dh' = -\tfrac{2}{3}h^3\left(\frac{dQ}{dz}\right),$$

also
$$\left(\frac{dQ}{dz}\right) = (m+n)\,\mu + (m-n)\left\{\lambda + \left(\frac{d\sigma_3}{dz}\right)\right\},$$

whence putting

$$E = \frac{m-n}{m+n}, \tag{6}$$

we obtain

$$G_1 = -\tfrac{4}{3}nh^3\{\mu + E(\lambda+\mu) + EA_1/2n\}. \tag{7}$$

Similarly,
$$G_2 = \quad \tfrac{4}{3}nh^3\{\lambda + E(\lambda+\mu) + EA_1/2n\}. \tag{8}$$

The value of A_1 is unknown, but according to the fundamental hypothesis A_1 does not contain any lower power of h than the first, whence the terms in A must be neglected, since the approximation is not carried farther than h^3.

Now by (2)

$$\lambda = \frac{d\sigma_1}{dz} = \frac{d^2u}{dzdx} = \frac{d\bar{\omega}_2}{dx} - \frac{d^2w}{dx^2},$$

also
$$\mu = \frac{d\sigma_2}{dz} = \frac{d^2v}{dzdy} = \frac{d\bar{\omega}_1}{dy} - \frac{d^2w}{dy^2}.$$

Since $S' = n\bar{\omega}_1'$ and $T' = n\bar{\omega}_2'$, it follows from the fundamental hypothesis that $\bar{\omega}_1$, $\bar{\omega}_2$, which are the values of $\bar{\omega}_1'$, $\bar{\omega}_2'$ at the middle surface, do not contain any lower power of h than h^2; accordingly the terms involving these quantities may be neglected when multiplied by h^3, whence the values of G_1, G_2 finally become

$$\left.\begin{aligned}G_1 &= \quad \tfrac{4}{3}nh^3\left(\frac{d^2w}{dy^2} + E\nabla^2 w\right),\\ G_2 &= -\tfrac{4}{3}nh^3\left(\frac{d^2w}{dx^2} + E\nabla^2 w\right),\end{aligned}\right\} \tag{9}$$

where ∇^2 has its usual meaning.

Again,

$$H_1 = -H_2 = -\tfrac{2}{3}nh^3\left(\frac{d\bar{\omega}_3}{dz}\right)$$

and
$$\frac{d\bar{\omega}_3}{dz}=\frac{d^2u}{dydz}+\frac{d^2v}{dxdz}=\frac{d\bar{\omega}_2}{dy}+\frac{d\bar{\omega}_1}{dx}-2\frac{d^2w}{dxdy},$$

so that to the above order of approximation

$$H_1=-H_2=\tfrac{4}{3}nh^3\frac{d^2w}{dxdy}. \tag{10}$$

Equations (9) and (10) completely determine the four couples.

9. If we expand u', v', w' in powers of h' and eliminate their differential coefficients with respect to z, the right-hand sides of the last three of (1) can be calculated by a similar process, and we shall obtain

$$\left.\begin{aligned}\frac{dN_2}{dx}+\frac{dN_1}{dy}&=\quad 2\rho h(\ddot{w}-Z)+\tfrac{1}{3}\rho h^3E\nabla^2\ddot{w},\\ \frac{dG_1}{dy}+\frac{dH_1}{dx}+N_1&=\quad\tfrac{2}{3}\rho h^3\frac{d\ddot{w}}{dy},\\ \frac{dG_2}{dx}+\frac{dH_2}{dy}-N_2&=-\tfrac{2}{3}\rho h^3\frac{d\ddot{w}}{dx}.\end{aligned}\right\} \tag{10, A}$$

Substituting the values of the couples from (9) and (10) in the last two, and the resulting values N_1 and N_2 in the first, we obtain the following equation of motion, viz.:

$$\tfrac{4}{3}nh^2(1+E)\nabla^4w+2\rho(\ddot{w}-Z)+\tfrac{1}{3}\rho h^2(E-2)\nabla^2\ddot{w}=0. \tag{11}$$

If $Z=0$ the last term must be omitted, since it gives rise to a term h^4 in the period, and (11) becomes

$$\frac{4mnh^2}{3(m+n)}\nabla^4w+\rho\ddot{w}=0. \tag{12}$$

The motion represented by this equation consists of a displacement which is perpendicular to the middle surface; moreover, the motion does not involve any extension or contraction of this surface, for the extensional strains are given by the first, second and sixth of (2), and when u and v are zero these strains are also zero. Vibrations of this kind are sometimes called *lateral vibrations* on the ground that the displacement is perpendicular to the middle surface, and sometimes *flexural vibrations*, because they involve flexion or bending of this surface unaccompanied by extension. The periods of these vibrations are much longer than those which depend upon extension alone, and they constitute the most

important class of vibrations which a plane plate is capable of executing. They are fully discused in Vol. I, Chap. X, of Lord Rayleigh's Theory of Sound.

10. We must now consider the boundary conditions which hold good at a free edge.

It is obvious that the flexural couple about a free edge must vanish; hence if the free edge is parallel to the axis of y, $G_2 = 0$, and therefore by the second of (9),

$$\frac{d^2w}{dx^2} + E\nabla^2 w = 0.$$

It might also be thought that the torsional couple H and the normal shearing stress N must also vanish, in which case we should have three boundary conditions. This was Poisson's view, but Kirchhoff showed that only one additional equation was necessary, and it is now well known that Poisson's boundary conditions are erroneous.

The reduction of Poisson's boundary conditions depends upon the curious theorem that it is possible to apply a system of stresses to the edge of the plate without doing any work. By Stokes' theorem

$$\int\left(\frac{dH'}{dx}\,\delta w + H'\,\frac{d\delta w}{dx}\right)dx + \int\left(\frac{dH'}{dy}\,\delta w + H'\,\frac{d\delta w}{dy}\right)dy = 0,$$

the integration extending round the rectangle $OADB$.

If we apply to the side AD the stresses

$$N_2' = \frac{dH'}{dy}, \quad H_1' = H',$$

to the side DB the stresses

$$N_1' = \frac{dH'}{dx}, \quad H_2' = -H',$$

and to the sides BO, OA corresponding and opposite stresses, the preceding integral becomes

$$\int\left(N_1'\delta w - H_2'\,\frac{d\delta w}{dx}\right)dx + \int\left(N_2'\delta w + H_1'\,\frac{d\delta w}{dy}\right)dy = 0,$$

which shows that the work done by this system of stresses is zero. Such a system of stresses may therefore be applied or removed without interfering with the equilibrium or motion of the plate.

If, therefore, we suppose that the rectangle $OADB$, instead of being under the action of the remainder of the plate, is isolated, and that constraining stresses are applied to its edges, then it follows that if instead of the torsional couples H_1, $-H_1$ (since $H_2 = -H_1$) due to the action of contiguous portions of the plate, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\left.\begin{aligned}\mathfrak{H}_1 &= \quad H_1 + H', \\ \mathfrak{H}_2 &= -H_1 - H',\end{aligned}\right\} \tag{13}$$

the energy and state of strain will be unaltered, provided we apply in addition the stresses

$$\left.\begin{aligned}\mathfrak{N}_2 &= N_2 + \frac{dH'}{dy}, \\ \mathfrak{N}_1 &= N_1 + \frac{dH'}{dx},\end{aligned}\right\} \tag{14}$$

whence eliminating H' between the first and the second of (13) and (14) respectively, we obtain

$$\left.\begin{aligned}\mathfrak{N}_2 - \frac{d\mathfrak{H}_1}{dy} &= N_2 - \frac{dH_1}{dy}, \\ \mathfrak{N}_1 + \frac{d\mathfrak{H}_2}{dx} &= N_1 + \frac{dH_2}{dx},\end{aligned}\right\} \tag{15}$$

In these equations the Roman letters denote the values of the stresses due to contiguous portions of the plate, whilst the Old English letters denote the actual stresses applied to the boundary; whence the conditions to be satisfied along a free edge parallel to the axis of y are

$$N_2 - \frac{dH_1}{dy} = 0, \tag{16}$$

and along one parallel to the axis of x,

$$N_1 - \frac{dH_1}{dx} = 0. \tag{17}$$

Since the frequency of the flexural vibrations depends upon the thickness, the terms on the right-hand sides of the last two of (10, A) may be omitted in the values of N_1, N_2 furnished by these equations, and (16) becomes

$$\frac{d}{dx}\left\{\frac{d^2w}{dx^2} + \frac{3m-n}{2m}\frac{d^2w}{dy^2}\right\} = 0.$$

When the plate is bounded by any plane curve, the two boundary conditions may be obtained in a similar manner. The work is given on pp. 42 and 43 of my paper on plane plates (Proc. Lond. Math. Soc., Vol. XXI), and will be found to furnish the same results as those given by Lord Rayleigh (Theory of Sound, Vol. I, p. 297).

The preceding analysis gives the complete theory of the flexural vibrations of a plane plate.

Extensional Vibrations.

11. We must now consider the extensional vibrations of the plate. The value of T_1 is

$$T_1 = \int_{-h}^{h} \left\{ P + h'\left(\frac{dP}{dz}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2P}{dz^2}\right) \right\} dh' = 2hP + \tfrac{1}{3}h^3\left(\frac{d^2P}{dz^2}\right) \qquad (20)$$

Now
$$P = (m+n)\,\sigma_1 + (m-n)(\sigma_2 + \sigma_3).$$

Substituting the value of σ_3 from the first of (5) we obtain

$$P = 2n\{\sigma_1 + E(\sigma_1+\sigma_2)\} + EA_0. \qquad (21)$$

We therefore see that T_1 contains the term $2EA_0h$, and since by the fundamental hypothesis A_0 is (or may be) proportional to h^2, the value of T_1 cannot be obtained by *direct integration* unless the value of A_0 is known.

The value of T_1 can, however, be obtained by an indirect method. I have shown in my paper on Plane Plates* that the various constituents of the equation which expresses the Principle of Virtual Work can be completely calculated as far as the terms proportional to h^3 without knowing the values of the quantities A_0, A_1, A_2 or the values of $\bar{\omega}_1$ and $\bar{\omega}_2$; and by working out this equation by the ordinary methods of the Calculus of Variations the correct values of T_1, T_2 and also M can be obtained as far as the term involving h^3. The problem is therefore capable of being completely solved to the order of approximation adopted, but for the details of the analysis the reader is referred to my paper. In the present article I shall only consider those portions of the extensional terms which are proportional to the *first* power of the thickness.

Let
$$\mathfrak{A} = \sigma_1 + E(\sigma_1+\sigma_2), \quad \mathfrak{B} = \sigma_2 + E(\sigma_1+\sigma_2), \qquad (22)$$

where σ_1, σ_2 denote the values of the extensions of a point *on* the middle surface;

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* Proc. Lond. Math. Soc., Vol. XXI, p. 33.

also let $\bar{\omega}$ denote the value of $\bar{\omega}_3'$ at the middle surface. Then from (20), (21) and (22) it follows that the value of the term in T_1 which is proportional to h is $4nh\mathfrak{A}$; and by a similar process we can find the values of T_2 and M to a first approximation. We thus obtain

$$T_1 = 4nh\mathfrak{A}, \quad T_2 = 4nh\mathfrak{B}, \quad M = 2nh\bar{\omega},$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (22) and

$$\sigma_1 = \frac{du}{dx}, \quad \sigma_2 = \frac{dv}{dy}, \quad \bar{\omega} = \frac{dv}{dx} + \frac{du}{dy}. \tag{23}$$

Accordingly the first two of (1) become

$$\left.\begin{aligned} \rho(\ddot{u} - X) &= 2n\left(\frac{d\mathfrak{A}}{dx} + \tfrac{1}{2}\frac{d\bar{\omega}}{dy}\right), \\ \rho(\ddot{v} - Y) &= 2n\left(\frac{d\mathfrak{B}}{dy} + \tfrac{1}{2}\frac{d\bar{\omega}}{dx}\right). \end{aligned}\right\} \tag{24}$$

The boundary conditions at a free edge are that T_1, T_2 and M should vanish there.

The motion which is represented by these equations consists of a displacement parallel to the middle surface, but there is no displacement perpendicular to this surface. It also follows from (23) that the motion involves extension of the middle surface, but it does not involve flexion, for the terms upon which flexion depends are zero, when w, the normal displacement, is zero.

From the preceding analysis it follows that the vibrations which a plane plate is capable of executing consist of two independent types, viz. flexural vibrations unaccompanied by extension, and extensional ones unaccompanied by flexion.

12. The reader who compares §§7 to 11 with Chapters XIX and XX of Mr. Love's treatise cannot, I think, fail to be struck with the simplicity and elegance of the above method; moreover, the complete theory of the extensional vibrations as far as the terms involving h^3 can also be obtained by employing the Calculus of Variations, which Mr. Love's method cannot apparently effect. At any rate he has not attempted any such theory in his book.

Energy of the Plate.

13. I have shown on page 46 of my paper on Plane Plates that the potential energy consists of three distinct sets of terms. The first term is a quadratic

function of the extensional strains σ_1, σ_2, $\bar{\omega}$ multiplied by the thickness. The second term is a similar function of the terms on which flexion depends, multiplied by the cube of the thickness. The third term is a quadratic function of the extensional strains and their differential coefficients, which is also multiplied by the cube of the thickness. To obtain the complete expression for the potential energy involves nearly three octavo pages of work which I do not propose to reproduce, as the reader who is curious upon the point can consult my paper, but the first two terms can be easily obtained by the preceding methods.

If W be the potential energy of the element $2hdS$, we have by the ordinary formula

$$W = \tfrac{1}{2}\int_{-h}^{h} \{(m+n)(\sigma_1' + \sigma_2' + \sigma_3')^2 + n\,[\bar{\omega}_1'^2 + \bar{\omega}_2'^2 + \bar{\omega}_3'^2 - 4(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1')]\}\,dh'.$$

By the fundamental hypothesis the terms involving $\bar{\omega}_1'$, $\bar{\omega}_2'$ are proportional to h^5 and are therefore to be neglected. The term involving h, upon which the purely extensional vibrations depend, can be easily shown to be

$$2nh\,\{\sigma_1^2 + \sigma_2^2 + E(\sigma_1 + \sigma_2)^2 + \tfrac{1}{2}\bar{\omega}^2\},$$

where σ_1, σ_2, $\bar{\omega}$ are given by (23), whilst the term in h^3 which depends upon w is

$$\tfrac{2}{3}nh^3\left\{\left(\frac{d^2w}{dx^2}\right)^2 + \left(\frac{d^2w}{dy^2}\right)^2 + E\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right)^2 + 2\left(\frac{d^2w}{dxdy}\right)^2\right\}.$$

Although the term in the potential energy which depends upon bending is multiplied by h^3, whilst that which depends upon extension is multiplied by h, the former term is the most important. One reason of this is, that the gravest notes yielded by a plate depend upon flexural vibrations, whilst those depending on extension are of high pitch. Also, it is far easier to bend a thin plate than to stretch it; consequently, if any system of forces were applied to the edges of the plate, the work done by the forces would be almost entirely concentrated in a deformation involving flexion. The extensional part of the deformation would be hardly appreciable in the case of a glass or metal plate; it would only become important in the case of highly extensible substances, such as india-rubber.

*Cylindrical Shells.**

14. The deformation of a naturally curved plate, which will be called a shell, is a problem of greater difficulty, because the vibrations are not in general separable into two distinct types one of which depends upon the normal dis-

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* See Phil. Trans., 1890, p. 438.

placement whilst the other depends upon the tangential displacement. The quantities which define the state of strain consist as before of two distinct sets of terms, viz. quantities upon which the extension of the middle surface depends, and quantities upon which the bending depends. The potential energy, unlike that of a plane plate, consists of four distinct sets of terms,* viz. (i) a term proportional to the thickness multiplied by a quadratic function of the extensions of the middle surface; (ii) a term proportional to the cube of the thickness multiplied by a quadratic function of quantities upon which pure flexion depends; (iii) a term proportional to the cube of the thickness multiplied by the product of the flexural and extensional terms; (iv) a term proportional to the cube of the thickness multiplied by a quadratic function of quantities upon which extension depends. The correct expression for the potential energy of a circular cylinder is given by equation (24), page 443, of my paper in the Philosophical Transactions for 1890, and the values of the various quantities involved are given on pp. 440 and 441.

15. In the figure to §2, let OA be the generator at any point O of a right circular cylindrical shell, and let OB be the circular section through O, whilst OC is the normal. Let u, v, w be the displacements of O along OA, OB and OC respectively. Let the unaccented letters denote the values of the different quantities at O, and the accented letters at a point P' of which O is the projection.

We shall commence by calculating the values of the four couples. The six equations connecting the strains and displacements are

$$\left.\begin{aligned}
\sigma_1' &= \frac{du'}{dz}, \\
\sigma_2' &= \frac{1}{r}\left(\frac{dv'}{d\phi} + w'\right), \\
\sigma_3' &= \frac{dw'}{dr}, \\
\bar{\omega}_1' &= \frac{dv'}{dr} - \frac{v'}{r} + \frac{1}{r}\frac{dw'}{d\phi}, \\
\bar{\omega}_2' &= \frac{dw'}{dz} + \frac{du'}{dr}, \\
\bar{\omega}_3' &= \frac{1}{r}\frac{du'}{d\phi} + \frac{dv'}{dz}.
\end{aligned}\right\} \qquad (1)$$

where r, ϕ, z are cylindrical coordinates.

* Phil. Trans., 1890, p. 443, equation (24).

Let λ, μ, p denote the values of $d\sigma_1/dr$, $d\sigma_2/dr$, $d\bar{\omega}_3/dr$ when $r=a$. The value of G_1 is

$$\left.\begin{aligned} G_1 &= -\int_{-h}^{h} Q'h'dh' \\ &= -\tfrac{2}{3}h^3\left(\frac{dQ}{dr}\right), \end{aligned}\right\} \tag{2}$$

whilst

$$\left.\begin{aligned} G_2 &= \int_{-h}^{h} P'\left(1+\frac{h'}{a}\right)h'dh' \\ &= \tfrac{2}{3}h^3\left\{\left(\frac{dP}{dr}\right)+\frac{P}{a}\right\}. \end{aligned}\right\} \tag{3}$$

Now

$$\begin{aligned} R' &= (m+n)\,\sigma_3' + (m-n)(\sigma_1'+\sigma_2') \\ &= (m+n)\,\sigma_3 + (m-n)(\sigma_1+\sigma_2) \\ &\qquad + \left\{(m+n)\left(\frac{d\sigma_3}{dr}\right) + (m-n)(\lambda+\mu)\right\}h' + \dots\dots, \end{aligned}$$

also
$$R' = A_0 + A_1h' + \dots\dots,$$

where by the fundamental hypothesis A_0, A_1 are proportional to h^2 and h respectively; whence

$$\left.\begin{aligned} \sigma_3 &= \frac{A_0}{m+n} - E(\sigma_1+\sigma_2), \\ \left(\frac{d\sigma_3}{dr}\right) &= \frac{A_1}{m+n} - E(\lambda+\mu). \end{aligned}\right\} \tag{4}$$

The values of P' and Q' are

$$\begin{aligned} P' &= (m+n)\,\sigma_1' + (m-n)(\sigma_2'+\sigma_3'), \\ Q' &= (m+n)\,\sigma_2' + (m-n)(\sigma_3'+\sigma_1'). \end{aligned}$$

Expand these values of P', Q' in powers of h', and substitute the values of σ_3 and $(d\sigma_3/dr)$ from (4); it will then be found that when the resulting values of P, (dP/dr) and (dQ/dr) are substituted in (2) and (3), the terms involving A_0 and A_1 are proportional to some higher power of h than h^3, and may therefore be neglected. We thus obtain

$$G_1 = -\tfrac{4}{3}nh^3\{\mu + E(\lambda+\mu)\}, \tag{5}$$
$$G_2 = \tfrac{4}{3}nh^3\{\lambda + E(\lambda+\mu) + [\sigma_1 + E(\sigma_1+\sigma_2)]/a\}. \tag{6}$$

Also by a similar process it can be shown that

$$H_1 = -\tfrac{2}{3}nh^3\left(p + \frac{\bar{\omega}_3}{a}\right), \tag{7}$$

$$H_2 = \tfrac{2}{3}nh^3 p. \tag{8}$$

16. The quantities σ_1, σ_2, $\bar{\omega}_3$ are determined by the equations

$$\sigma_1 = \frac{du}{dz},\quad \sigma_2 = \frac{1}{a}\left(\frac{dv}{d\phi} + w\right),\quad \bar{\omega}_3 = \frac{1}{a}\frac{du}{d\phi} + \frac{dv}{dz}, \tag{10}$$

and we must now calculate the values of λ, μ, p.

From the first and fifth of (1) we obtain

$$\lambda = \left(\frac{d\sigma_1}{dr}\right) = \frac{d\bar{\omega}_2}{dz} - \frac{d^2w}{dz^2}.$$

Now by the fundamental hypothesis $\bar{\omega}_2$ is proportional to h^2, accordingly the term $d\bar{\omega}_2/dz$ when multiplied by h^3 is proportional to h^5 and may therefore be neglected. We thus obtain

$$\lambda = -\frac{d^2w}{dz^2}. \tag{11}$$

Similarly from the second and fourth of (1) we obtain

$$\mu = -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2). \tag{12}$$

Lastly,

$$p = \left(\frac{d\bar{\omega}_3}{dr}\right) = \frac{1}{a}\left(\frac{d^2u}{drd\phi}\right) + \left(\frac{d^2v}{drdz}\right) - \frac{1}{a^2}\frac{du}{d\phi}.$$

Substituting the values of (dv/dr), (du/dr) from the fourth and fifth of (1), and recollecting that the terms in $\bar{\omega}_1$, $\bar{\omega}_2$ are to be neglected when multiplied by h^3, we obtain

$$p = -\frac{2}{a}\frac{d^2w}{dzd\phi} + \frac{1}{a}\frac{dv}{dz} - \frac{1}{a^2}\frac{du}{d\phi}. \tag{13}$$

Equations (11), (12) and (13) combined with (5), (6), (7) and (8) completely determine the couples to the above order of approximation.

It is very important to notice that the complete values of the couples contain certain terms depending on the extension of the middle surface, as well as terms depending on the bending.

Inextensible Deformations.

17. In a large and important number of problems, the extension of the middle surface may be neglected. In this case σ_1, σ_2, $\bar{\omega}_3$ are zero, and the displacements are connected together by the three equations which are obtained by equating the right-hand sides of (10) to zero; accordingly, since $du/dz = 0$, we have

$$G_1 = \frac{4nh^3}{3a^2}(1+E)\left(\frac{d^2w}{d\phi^2} + w\right), \tag{14}$$

$$G_2 = -\frac{4nh^3}{3a^2}E\left(\frac{d^2w}{d\phi^2} + w\right), \tag{15}$$

$$H_1 = -H_2 = \frac{4nh^3}{3a}\left(\frac{d^2w}{dzd\phi} - \frac{dv}{dz}\right). \tag{16}$$

18. We have now obtained the materials which we require for solving the problem of the flexural vibrations of an infinitely long cylindrical shell which is bent about a generator.

The equations of motion of the cylinder are

$$\left.\begin{aligned} \frac{1}{a}\frac{dT}{d\phi} + \frac{N}{a} &= \nu\ddot{v}, \\ \frac{1}{a}\frac{dN}{d\phi} - \frac{T}{a} &= \nu\ddot{w}, \\ \frac{1}{a}\frac{dG}{d\phi} + N &= 0. \end{aligned}\right\} \tag{17}$$

where $\nu = 2h\sigma$, σ being the density. Also the condition of inextensibility is

$$\frac{dv}{d\phi} + w = 0. \tag{18}$$

Substituting the value of G from (14) in the last of (17), we obtain the value of N; and if we substitute this value of N in the first two of (17) and then eliminate T, we shall obtain the equation

$$\frac{4mnh^2}{3\rho a^4(m+n)}\left(\frac{d^3}{d\phi^3} + \frac{d}{d\phi}\right)^2 v + \frac{d^2\ddot{v}}{d\phi^2} - \ddot{v} = 0, \tag{19}$$

whence putting

$$v = A\epsilon^{\iota pt + \iota s\phi},$$

we obtain

$$p^2 = \frac{4mnh^2s^2(s^2-1)^2}{3\rho a^4(m+n)(s^2+1)}, \tag{20}$$

which determines the period of the vibrations.

19. Equation (20) is discussed in Lord Rayleigh's Theory of Sound, Vol. I, Chap. X, and also by Prof. H. Lamb (Proc. Lond. Math. Soc., Vol. XIX, p. 365). When the cylinder is complete, s is any integer, unity excluded; but if the cross-section of the cylinder consists of a circular arc of length $2a\alpha$, s will not be an integer. In the latter case the values of s must be determined as follows: Along the two generators which form the free edges, T, N and G must vanish; in other words, if the middle point of the arc be taken as the origin of ϕ, these stresses must vanish when $\phi = \pm\alpha$. The complete solution of (19) involves six arbitrary constants, and the boundary conditions give six equations which will enable the six constants to be eliminated; and the resulting equation combined with (20) will enable s to be eliminated and an equation to be obtained which will determine p in terms of known quantities.

20. The preceding solution raises a curious point which excited a good deal of controversy in England about five years ago. Although the displacement parallel to the generators is expressly assumed to be zero, yet in this case there is a flexural couple G_2 about each circular section whose value by virtue of (15) is

$$G_2 = -\frac{4nh^3}{3a^2}E\left(\frac{d^2w}{d\phi^2}+w\right), \tag{21}$$

the extensibility being neglected.

Let us now suppose that the cylinder is of finite length $2z$; we have in this case to satisfy the boundary conditions at the two circular edges, and one of these conditions is that $G_2 = 0$, because the flexural couple about a circular section must vanish at a free edge. This requires that

$$\frac{d^2w}{d\phi^2} + w = 0,$$

whence at the edge

$$w = A\cos\phi + B\sin\phi$$

for all values of ϕ.

Now it is impossible to satisfy this condition for the most general type of non-extensional vibrations, and hence the solution fails; in other words, when

the cylinder is of finite length, purely flexural vibrations unaccompanied by extension of the middle surface are impossible.

This was the controversy raised by Mr. Love in 1888; but on examining the question it appeared to me that it might be possible for some extension to take place which was confined to points in the immediate neighborhood of the free edge, and which was inappreciable at points whose distance from the free edge was large in comparison with the thickness. But in order to satisfactorily deal with the question I found it necessary to obtain the correct values of all the quantities involved as far as terms proportional to the cube of the thickness, *on the supposition that the middle surface underwent extension.* This required a long and difficult investigation by means of the Calculus of Variations, which is given at length in my paper (Phil. Trans., 1890, p. 433); but I was finally enabled to establish the correctness of my supposition by solving the following problem.

21. Let a very long, heavy cylindrical shell, whose cross-section is a semicircle, be suspended by vertical bands attached to its straight edges, so that its axis is horizontal. It is required to investigate the state of strain produced by its own weight.

I found that if R denote the change of curvature along a circular section, so that

$$R = -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right),$$

w being the displacement along the radius, and σ_2 is the extension of the middle surface along the circular section, so that

$$\sigma_2 = \frac{1}{a}\left(\frac{dv}{d\phi} + w\right),$$

then

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a\,(\frac{1}{2}\pi - \phi\sin\phi - \cos\phi)}{h^2\{\frac{1}{2}\pi - \cos\phi + \frac{3}{2}E(\frac{1}{2}\pi - \phi\sin\phi - \cos\phi)\}},$$

where ϕ is measured from the lowest point.

Since the numerator of this fraction is an even function of ϕ, it does not change sign with ϕ; also the numerator is always positive between the limits $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, and its maximum value occurs when $\phi = 0$, and is equal to

$\frac{1}{2}\pi - 1$, and its minimum value occurs when $\phi = \frac{1}{2}\pi$ and is equal to zero. We therefore see that when $\phi = 0$,

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a}{h^2},$$

and when $\phi = \frac{1}{2}\pi$,

$$\frac{R}{\sigma_2} = \frac{E}{a}.$$

Since the thickness of the shell is supposed to be small compared with its radius, it follows that the change of curvature is large compared with the extension of the middle surface except when $a(\frac{1}{2}\pi - \phi)$ is comparable with h, i. e. in the neighborhood of the straight edges of the shell; and therefore at all points of the shell whose distances from the edges are large in comparison with the thickness, the terms depending on the product of the change of curvature and the cube of the thickness, i. e. the terms upon which the bending depends, are of the same order as those depending upon the product of the middle surface and the thickness; but at points whose distances from the edge are comparable with the thickness of the shell, the extension of the middle surface is of the same order as the change of curvature, and therefore the terms depending upon the product of the change of curvature and the cube of the thickness are small in comparison with those depending upon the product of the extension and the thickness.

22. Now we have already pointed out in §14 that the potential energy consists of four distinct terms, the first, third and fourth of which involve extension, whilst the second depends upon pure flexion unaccompanied by extension; and the preceding result shows that the second term is by far the most important.

Under these circumstances we are justified in concluding that the solution given in §18 is substantially correct, and that any extension which takes place is practically confined to points in the neighborhood of a free edge, and is practically negligible.

23. By means of the fundamental hypothesis that the three stresses R', S', T' may be treated as zero, the value of the potential energy when the middle surface undergoes no extension can easily be shown to be

$$W = \frac{2}{3}nh^3\{\lambda^2 + \mu^2 + E(\lambda + \mu)^2 + \frac{1}{2}p^2\}.$$

Substituting the values of μ, p from (12) and (13) and taking into account the conditions of inextensibility—see equation (10)—which require that $\lambda = 0$, this may be expressed in the form

$$W = \frac{4nh^3}{3a^2}\left\{\frac{m}{(m+n)a^2}\left(\frac{d^2w}{d\phi^2}+w\right)^2+\left(\frac{d^2w}{dzd\phi}-\frac{dv}{dz}\right)^2\right\}. \quad . \qquad (22)$$

From the second of (10) this may be expressed in terms of v if desired.

The conditions of inextensibility (10) require that

$$u = Ua,\ v = -z\frac{dU}{d\phi}+V,\ w = z\frac{d^2U}{d\phi^2}-\frac{dV}{d\phi}, \qquad (23)$$

and since U and V must necessarily be periodic functions of ϕ, provided the cylinder is complete, we may express them in series of the form

$$U = B_s\epsilon^{\iota s\phi},\quad V = A_s\epsilon^{\iota s\phi},$$

where A and B are complex functions of the time. The potential energy of a cylindrical shell of length $2z$ may be obtained by substituting the above value of u, v, w, U, V in (22) and integrating throughout its length. Writing out the full values of u, v, w in real quantities, we get, as shown by Lord Rayleigh,*

$$u = -s^{-1}a\Sigma(B_s\sin s\phi + B_s'\cos s\phi),$$
$$v = \Sigma\{(A_sa + B_sz)\cos s\phi - (A_s'a + B_s'z)\sin s\phi\},$$
$$w = \Sigma\{s(A_sa + B_sz)\sin s\phi + s(A_s'a + B_s'z)\cos s\phi\}.$$

The value of the potential energy can now be obtained in terms of the A's and B's, whilst the kinetic energy is obtained from the equation

$$T = h\rho\int\int(\dot{u}^2+\dot{v}^2+\dot{w}^2)\,ad\phi dz.$$

Having evaluated T and W, the motion can be obtained by Lagrange's equations, and it will be found that the vibrations consist of two distinct types. The first type depends upon the A's, and are those which we have already investigated; they involve pure flexion, and are such as would be produced if every circular section were slightly deformed into an equal and similar curve. The second type involve torsion as well as flexion, and each circular section is twisted into a sinuous curve about its mean position.

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* Proc. Roy. Soc., Vol. XLV, p. 105.

The periods of the latter class of vibrations are given by the equation

$$p'^2 = \frac{4mnh^2 s^2 (s^2 - 1)^2}{3\rho a^4 (m+n)(s^2+1)} \cdot \frac{1 + 3a^2(m+n)/ms^2l^2}{1 + 3a^2/s^2(s^2+1)l^2},$$

where $s = 2, 3, \ldots$.

For the further discussion of these vibrations we must refer to Lord Rayleigh's paper; we notice that if the cylinder be at all long in proportion to its diameter, the difference between the periods of the two types of vibrations becomes small.

24. Before leaving this branch of the subject, it may be well to point out that Kirchhoff's four boundary conditions apply to the case of a cylindrical shell as well as to a plane plate.* The principles upon which the theory rests are identical in both cases, although the mathematical details are different.

Along a free edge which is a circular section, the conditions are

$$T_1 = 0, \qquad G_2 = 0,$$
$$M_2 = H_1/a, \quad N_2 = \frac{1}{a}\frac{dH_1}{d\phi},$$

whilst along a generator the conditions are

$$T_2 = 0, \quad M_1 = 0,$$
$$G_1 = 0, \quad N_1 + \frac{dH_2}{dz} = 0.$$

The proof depends upon the fact that the integral

$$\int\left(\frac{dH'}{d\phi}\delta w + H'\frac{d\delta w}{d\phi}\right)d\phi + \int\left(\frac{dH'}{dz}\delta w + H'\frac{d\delta w}{dz}\right)dz = 0,$$

when taken round a rectangle bounded by four lines of curvature, vanishes by Stokes' theorem.

Extensional Vibrations.

25. The potential energy contains a term proportional to the thickness multiplied by a quadratic function of the extensions. The value of this term is

$$W = 2nh\{\sigma_1^2 + \sigma_2^2 + E(\sigma_1 + \sigma_2)^2 + \tfrac{1}{2}\bar{\omega}_3^2\},$$

which can easily be established by the methods of the present paper.

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* See p. 453 of my paper, Phil. Trans., 1880.

The values of T_1, T_2, M_1 and M_2, so far as they depend on h alone and not on h^3, are

$$T_1 = 4nh\mathfrak{A}, \quad M_1 = 2nh\bar{\omega}_3,$$
$$T_2 = 4nh\mathfrak{B}, \quad M_2 = 2nh\bar{\omega}_3,$$

where $$\mathfrak{A} = \sigma_1 + E(\sigma_1 + \sigma_2), \quad \mathfrak{B} = \sigma_2 + E(\sigma_1 + \sigma_2).$$

The equations of motion are

$$\rho\ddot{u} = 2n\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\bar{\omega}_3}{d\phi}\right) + \rho X,$$
$$\rho\ddot{v} = 2n\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\bar{\omega}_3}{dz}\right) + \rho Y,$$
$$\rho\ddot{w} = -2n\mathfrak{B}/a + \rho Z,$$

whilst the boundary conditions at a free edge are that T_1, T_2, M_1, M_2 should vanish there.

The solution of these equations when no bodily forces act has been discussed by Mr. Love,* and also by Lord Rayleigh,† to whose papers the reader is referred. The vibrations are of high pitch, and are of mathematical rather than physical interest.

26. Before passing on, it may be as well to take this opportunity of replying to a criticism which Mr. Love has made in the notes to pp. 238 and 262 of Volume II of his Treatise on Elasticity. He observes that the value of T_2 given by the second of my equations (44) (Phil. Trans., 1890, p. 450) does not furnish the same value of T_2 as the equations given at the bottom of p. 456, and not being able to understand this, he imagines that he has detected some error in my work. The explanation is simple. In the investigation by which equations (44) were obtained, the middle surface is expressly supposed to undergo extension, whereas in the problem discussed on p. 456 no extension is supposed to take place. The two cases require totally different treatment, and are therefore not parallel; consequently Mr. Love's so-called test is nugatory. If there were any error in the value of T_2, the equations obtained by substituting the values of the stresses given by (43) and (44) in the first three of (11) would not reproduce equations (45), (46) and (47), as they actually do.

The point where Mr. Love has gone astray is the following:

Let F be any function of u, v and their differential coefficients with respect to x and y. The condition that the integral

$$\iint \delta F dxdy$$

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* Phil. Trans., 1888, pp. 538, etc.

† Proc. Roy. Soc., Vol. XLV, p. 443.

should vanish is obtained by integrating it by parts until it is reduced to the form

$$\iint (M\delta u + N\delta v)\,dx\,dy + \text{a line integral.}$$

Now if δu, δv are independent quantities, the surface integral gives rise to the equations

$$M = 0, \quad N = 0,$$

which must hold good at every point of the surface, and the line integral gives rise to similar equations which must hold good at the boundary. If, however, δu, δv are not independent but are connected together by some relation, as is the case when the surface undergoes no extension, the equations $M = 0$, $N = 0$, as well as those furnished by the line integral, are no longer true. When the relation is of the form

$$v = f(u,\ du/dx, \ldots.),$$

so that v is given explicitly in terms of u, we can substitute for δv in terms of δu and integrate the term $\iint N\delta v dx dy$ until it is reduced to the form

$$\iint N'\delta u dx dy + \text{ a line integral,}$$

and since δu is independent, the surface integral furnishes the single equation

$$M + N' = 0,$$

and the equations furnished by the line integral terms will also be different.

Instead of adopting the above-mentioned procedure, we may, if we please, introduce an indeterminate multiplier, in which case we shall obtain the two equations

$$M + p = 0, \quad N + q = 0,$$

where M and N have their former meanings and p and q are additional terms which depend upon the indeterminate multiplier. The elimination of this quantity must necessarily lead to an equation equivalent to $M + N' = 0$.

The failure to recognize the distinction between the two cases constitutes the error of Mr. Love's criticism.

*Spherical Shells.**

27. The theory of spherical shells can be worked out in exactly the same manner. In the figure to §2, let OA be a meridian and OB a parallel of lati-

* Phil. Trans. 1890, p. 463.

tude. The six strains are given by the equations

$$\left.\begin{aligned}
\sigma_1' &= \frac{1}{r}\left(\frac{du'}{d\theta} + w'\right), \\
\sigma_2' &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{dv'}{d\phi} + u'\cot\theta + w'\right), \\
\sigma_3' &= \frac{dw'}{dr}, \\
\bar{\omega}_1' &= \frac{1}{r\sin\theta}\frac{dw'}{d\phi} + \frac{dv'}{dr} - \frac{v'}{r}, \\
\bar{\omega}_2' &= \frac{du'}{dr} - \frac{u'}{r} + \frac{1}{r}\frac{dw'}{d\theta}, \\
\bar{\omega}_3' &= \frac{1}{r}\left(\frac{dv'}{d\theta} - v'\cot\theta + \frac{1}{\sin\theta}\frac{du'}{d\phi}\right).
\end{aligned}\right\} \quad (1)$$

By direct integration we find

$$G_1 = -\int_{-h}^{h} Q'\left(1 + \frac{h'}{a}\right) h'\,dh',$$

$$= -\tfrac{2}{3}h^3\left\{\left(\frac{dQ}{dr}\right) + \frac{Q}{a}\right\}, \quad (2)$$

$$G_2 = \tfrac{2}{3}h^3\left\{\left(\frac{dP}{dr}\right) + \frac{P}{a}\right\}, \quad (3)$$

$$H_1 = -H_2 = -\tfrac{2}{3}nh^3\left\{\left(\frac{d\bar{\omega}_3}{dr}\right) + \frac{\bar{\omega}_3}{a}\right\}. \quad (4)$$

Putting as before

$$\mathfrak{A} = \sigma_1 + E(\sigma_1 + \sigma_2), \quad \mathfrak{B} = \sigma_2 + E(\sigma_1 + \sigma_2), \quad (5)$$

$$\mathfrak{E} = \lambda + E(\lambda + \mu), \quad \mathfrak{F} = \mu + E(\lambda + \mu), \quad (6)$$

equations (2), (3) and (4) become

$$\left.\begin{aligned}
G_1 &= -\tfrac{4}{3}nh^3\left(\mathfrak{F} + \frac{\mathfrak{B}}{a}\right), \quad G_2 = \tfrac{4}{3}nh^3\left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right), \\
H_1 &= -H_2 = -\tfrac{2}{3}nh^3\left(p + \frac{\bar{\omega}_3}{a}\right),
\end{aligned}\right\} \quad (7)$$

where λ, μ, p denote the values of $d\sigma_1/dr$, $d\sigma_2/dr$ and $d\bar{\omega}_3/dr$ when $r = a$.

Recollecting that R', $\bar{\omega}_1'$ and $\bar{\omega}_2'$ are to be treated as zero by virtue of the fundamental hypothesis, we readily obtain to the required order of approximation,

$$\left.\begin{aligned}
\lambda &= -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2), \\
\mu &= -\frac{1}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2w}{d\phi^2} + \cot\theta\frac{dw}{d\theta} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2), \\
p &= \frac{2}{a^2\sin\theta}\left(\cot\theta\frac{dw}{d\phi} - \frac{d^2w}{d\theta d\phi}\right).
\end{aligned}\right\} \quad (8)$$

The proof of the above formulæ may be left to the reader, and he will find no difficulty in establishing them by means of the methods explained in the earlier part of this paper.

28. We shall now calculate the potential energy in the particular case in which the middle surface undergoes no extension.

The expression for W is

$$W = \tfrac{1}{2}\int\int\int_{-h}^{h}[(m+n)\,\Delta'^2 + n\{\bar{\omega}'^2 - 4(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1')\}](1+h'/a)^2 dh'dS, \quad (9)$$

where Δ' is the dilatation at the point $a+h'$, θ, ϕ; $\bar{\omega}'$ is written for $\bar{\omega}_3'$, and $\bar{\omega}_1'^2$, $\bar{\omega}_2'^2$ are omitted because they would give rise to terms proportional to h^5.

Since there is no extension, $\Delta = A_0/(m+n)$, and since A_0 is proportional to h^2, the only term in Δ' which it is necessary to retain is $h'(d\Delta/dr)$; we accordingly obtain

$$\tfrac{1}{2}(m+n)\int_{-h}^{h}\Delta'^2(1+h'/a)^2\,dh' = \frac{4n^2h^3}{3(m+n)}(\lambda+\mu)^2;$$

also since $\sigma_1 = \sigma_2 = 0$, we have

$$\sigma_1' = h'\lambda, \quad \sigma_2' = h'\mu,$$

so that

$$2n\int_{-h}^{h}\sigma_1'\sigma_2'(1+h'/a)^2\,dh' = \tfrac{4}{3}nh^3\lambda\mu.$$

In the same manner

$$2n\int_{-h}^{h}(\sigma_1' + \sigma_2')\,\sigma_3'(1+h'/a)^2\,dh = -\tfrac{4}{3}nh^3(\lambda+\mu)^2$$

and

$$\tfrac{1}{2}n\int_{-h}^{h}\bar{\omega}'^2(1+h/a)^2\,dh' = \tfrac{1}{3}nh^3p^2.$$

Hence if W now denote the potential energy per unit of area,

$$W = \tfrac{4}{3}nh^3\{\lambda^2 + \mu^2 + E(\lambda+\mu)^2 + \tfrac{1}{2}p^2\}. \quad (10)$$

The values of λ, μ, p are given by (8), but the extensional terms must be put equal to zero.

The complete expression for the potential energy as far as h^3, when the middle surface is supposed to undergo extension, is given by equation (16), p. 467 of my paper in the Phil. Trans., 1890. It will be observed that this expression (as in the case of a cylindrical shell) consists of four distinct sets of terms: (i) a term proportional to h multiplied by a quadratic function of the extensions, whose value is

$$2nh\{\sigma_1^2 + \sigma_2^2 + E(\sigma_1+\sigma_2)^2 + \tfrac{1}{2}\bar{\omega}^2\}; \quad (11)$$

(ii) a term proportional to h^3, which is multiplied by a quadratic function of quantities upon which pure bending depends; (iii) a term proportional of h_3 multiplied by the products of extensional and flexural terms; (iv) a term proportional to h^3 multiplied by a quadratic function of extensional terms. It will not, however, be necessary to consider the last two terms.

Flexural Vibrations.

29. Let ρ_1, ρ_2 be the radii of curvature of the deformed surface along and perpendicular to a meridian; then

$$\left.\begin{aligned} \frac{1}{\rho_1} - \frac{1}{a} &= -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right), \\ \frac{1}{\rho_2} - \frac{1}{a} &= -\frac{1}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta} + w\right). \end{aligned}\right\} \tag{12}$$

These formulæ are probably well known in America, but if a proof is desired one will be found on p. 477 of my paper. We notice from (8) that when extension does not take place, λ and μ are proportional to the changes of curvature along and perpendicular to a meridian.

The flexural vibrations of a spherical bowl have been investigated by Lord Rayleigh.* The conditions of inextensibility are given by the first, second and sixth of (1) and are

$$\left.\begin{aligned} &\frac{du}{d\theta} + w = 0, \\ &\frac{dv}{d\phi} + u\cos\theta + w\sin\theta = 0, \\ &\frac{du}{d\phi} - v\cos\theta + \sin\theta\frac{dv}{d\theta} = 0. \end{aligned}\right\} \tag{13}$$

Let $\chi = \log\tan\frac{1}{2}\theta$, then since

$$\sin\theta\frac{d}{d\theta} = \frac{d}{d\chi},$$

we find upon eliminating w that

$$\frac{d}{d\phi}\left(\frac{u}{\sin\theta}\right) + \frac{d}{d\chi}\left(\frac{v}{\sin\theta}\right) = 0,$$
$$\frac{d}{d\phi}\left(\frac{v}{\sin\theta}\right) - \frac{d}{d\chi}\left(\frac{u}{\sin\theta}\right) = 0,$$

consequently u cosec θ and v cosec θ both satisfy an equation of the form

$$\frac{d^2V}{d\phi^2} + \frac{d^2V}{d\chi^2} = 0.$$

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* Proc. Lond. Math. Soc., Vol. XIII, p. 4.

In the case of a spherical bowl, all the quantities are necessarily periodic functions of ϕ, and we may therefore suppose that ϕ enters in the form of the factor $\varepsilon^{\iota s\phi}$ where s is an integer; whence

$$\frac{d^2 V}{d\chi^2} - s^2 V = 0,$$

giving

$$V = A \tan^s \tfrac{1}{2}\theta + B \cot^s \tfrac{1}{2}\theta.$$

If θ is measured from the pole, we must have $B = 0$, otherwise V would become infinite when $\theta = 0$; we may therefore take

$$\left.\begin{aligned} u &= -\Sigma A_s \varepsilon^{\iota s\phi} \sin\theta \tan^s \tfrac{1}{2}\theta, \\ v &= \iota\Sigma A_s \varepsilon^{\iota s\phi} \sin\theta \tan^s \tfrac{1}{2}\theta, \\ w &= \Sigma A_s \varepsilon^{\iota s\phi} (s + \cos\theta) \tan^s \tfrac{1}{2}\theta, \end{aligned}\right\} \tag{14}$$

where $s = 2, 3, 4, \ldots$, and A_s is a complex function of the time. From these equations it can be shown by (12) that

$$\frac{1}{\rho_1} - \frac{1}{a} = -\Sigma \frac{A_s (s^3 - s)\, \varepsilon^{\iota s\phi} \tan^s \tfrac{1}{2}\theta}{a \sin^2\theta} = -\left(\frac{1}{\rho_2} - \frac{1}{a}\right), \tag{15}$$

also by the last of (8),

$$p = -2\iota\Sigma \frac{A_s (s^3 - s)\, \varepsilon^{\iota s\phi} \tan^s \tfrac{1}{2}\theta}{a \sin^2\theta}. \tag{16}$$

The value of A_s may be supposed to be real without loss of generality; accordingly, rejecting the imaginary part, we obtain from (10), (12), (15) and (16),

$$W = \frac{8\pi n h^3}{3a^2} \int_0^\alpha \operatorname{cosec}^3\theta\, \Sigma A_s^2 s^2 (s-1)^2 \tan^{2s} \tfrac{1}{2}\theta\, d\theta,$$

where α is the angle of the bowl.

Lord Rayleigh has applied the above expression for W to determine the notes produced by a hemispherical bell, but he has not attempted to take into account the conditions which must be satisfied at a free edge. Now one of these conditions is that the flexural couple G_2 should vanish there, where

$$\begin{aligned} G_2 &= \tfrac{4}{3} n h^3 \{\lambda + E(\lambda + \mu)\} \\ &= \tfrac{4}{3} n h^3 \left(\frac{1}{\rho_1} - \frac{1}{a}\right). \end{aligned}$$

From (15) we see that G_2 cannot vanish at a free edge, and consequently a spherical bell whose edge is free cannot vibrate in this manner if the middle surface is supposed to remain *absolutely* inextensible during the motion; but since the complete value of G_2 contains terms depending on the extensibility, it

would be possible to satisfy the boundary conditions if extension were supposed to take place. From the corresponding results in the case of a cylindrical shell, we should expect that some local extension must necessarily take place in the immediate neighborhood of a free edge; but at points whose distance from this edge is at all large compared with the thickness the extension becomes negligible, so that the solution given by Lord Rayleigh is for all practical purposes a sufficiently accurate one.

Extensional Vibrations.

30. We have already pointed out that the potential energy contains a term proportional to the thickness which depends upon the extension alone. This term leads to the following equations, viz.

$$T_1 = 4nh\mathfrak{A}, \quad T_2 = 4nh\mathfrak{B}, \quad M_1 = M_2 = 2nh\bar{\omega}, \tag{17}$$

and the equations of motion are

$$\left.\begin{aligned} \rho\ddot{u} &= \frac{2n}{a\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\bar{\omega}}{d\phi}\right\} + \rho X, \\ \rho\ddot{v} &= \frac{2n}{a\sin\theta}\left\{\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d}{d\theta}(\bar{\omega}\sin\theta) + \tfrac{1}{2}\bar{\omega}\cos\theta\right\} + \rho Y, \\ \rho\ddot{w} &= -\frac{2n}{a}(\mathfrak{A} + \mathfrak{B}) + \rho Z. \end{aligned}\right\} \tag{18}$$

Equations (17), which are approximate, may be deduced by direct integration, whilst (18) may be obtained by writing down the equations giving the motion of *translation* of an element. The terms involving the normal shearing stresses N_1, N_2 must be omitted because these quantities are proportional to the cube of the thickness. Having done this, we substitute the values of T_1, T_2 and M from (17).

The integration of these equations has been discussed by Mr. Love.

Collapse of a Boiler Flue.

31. A problem of great practical importance in engineering is that of the collapse of a boiler flue. The pressure of the smoke and the hot air from the furnace which passes through the flue is approximately equal to the atmospheric pressure, whilst the pressure of the steam within the boiler is considerably greater. Under these circumstances the flue shows a tendency to collapse when the difference between the pressures is sufficiently great.

The complete mathematical theory has not yet been made out, and the difficulty arises from the fact, that since the flue is subjected to an external pressure π_1, where π_1 is the pressure of the steam in the boiler, and an internal pressure π_2, where π_2 is the pressure of the hot air from the furnace, it is not permissible to suppose that R' may be treated as zero throughout the substance of the flue; for as we pass from the outer to the inner surface R' must vary from $-\pi_1$ to $-\pi_2$. The value of R' is

$$R' = A_0 + A_1 h' + \tfrac{1}{2} A_2 h'^2 + \ldots\ldots,$$

where h' is a small quantity, but the A's are not; we must therefore retain them in the expression for the flexural couple, and consequently the value of this quantity cannot be found unless the A's are known.

In a paper published in the Phil. Mag. for Sept. 1892 I fully discussed the difficulties of constructing a satisfactory theory. It seems allowable to assume that S and T may be treated as zero, in which case the value of the flexural couple G about a generating line can be shown to be

$$G = -\tfrac{2}{3} E A_1 h^3 - \frac{8mnh^3}{3a(m+n)}\left(\frac{A_0}{m+n} - \frac{w}{a} - \frac{1}{a}\frac{d^2w}{d\phi^2}\right),$$

the extension being neglected.

As we do not know anything about the A's, I assumed provisionally that

$$G = \frac{8h^3}{3a^2}\left(\frac{mn}{m+n} + \alpha\right)\left(\frac{d^2w}{d\phi^2} + w\right), \tag{1}$$

where α is some unknown function of the pressures π_1 and π_2, which vanishes when these quantities are zero.

The equations determining the small oscillations are

$$\frac{dT}{ds} + \frac{N}{\rho} = 2h\sigma v,$$

$$\frac{dN}{ds} - \frac{T}{\rho} + \pi_2\left(1 - \frac{h}{\rho}\right) - \pi_1\left(1 + \frac{h}{\rho}\right) = 2h\sigma\ddot{w},$$

$$\frac{dG}{ds} + N = 0,$$

where ρ is the radius of curvature of the *deformed* middle surface and σ the density. The condition of inextensibility is

$$\frac{dv}{d\phi} + w = 0; \tag{2}$$

also
$$\frac{1}{\rho} = \frac{1}{a} - \frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right)$$
and $ds = ad\phi$. The equations of motion may now be written

$$\frac{dT}{d\phi} + N = 2ha\sigma\ddot{v},$$

$$\frac{dN}{d\phi} - T + (\pi_2 - \pi_1)\,a - (\pi_2 + \pi_1)\,h + (\pi_2 - \pi_1)\left(\frac{d^2w}{d\phi^2} + w\right) = 2ha\sigma\ddot{w},$$

$$\frac{dG}{d\phi} + Na = 0.$$

Substituting the value of G from (1), eliminating T and N and taking account of (2), we finally obtain

$$\{Ia^{-1}(D^2 + 1) + \pi_1 - \pi_2\}(D^2 + 1)\,D^2v + 2ha\sigma\,(D^2 - 1)\,\ddot{v} = 0, \qquad (3)$$

where
$$I = \frac{8h^3}{3a^2}\left(\frac{mn}{m+n} + \alpha\right), \quad D = d/d\phi.$$

To solve (3) assume that
$$v \propto \varepsilon^{\iota pt + \iota s\phi},$$
where $s = 2, 3, 4, \ldots\ldots$, then
$$\{Ia^{-1}(s^2 - 1) - (\pi_1 - \pi_2)\}(s^2 - 1)\,s^2 = 2ha\sigma\,(s^2 + 1)\,p^2,$$

and therefore p will be real provided
$$\pi_1 - \pi_2 < I(s^2 - 1)/a.$$

The least value of the right-hand side occurs when $s = 2$, in which case

$$\begin{aligned}\pi_1 - \pi_2 &< 3I/a \\ &< \frac{8h^3}{a^3}\left(\frac{mn}{m+n} + \alpha\right).\end{aligned} \qquad (4)$$

If it were allowable to neglect α, the condition would become

$$\pi_1 - \pi_2 < \frac{8mnh^3}{a^3(m+n)}. \qquad (5)$$

Finite Bending of Thin Shells.

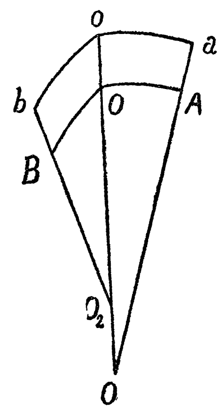
32. The problem of the finite bending of a thin shell presents difficulties which cannot yet be said to have been completely surmounted. When the deformation is small, the displacements of a point on the middle surface are small quantities whose squares and higher powers may be neglected in the cal-

culation of the strains; but when the deformation is finite the displacements may be large quantities. Since any deformation involves a change in the values of certain geometrical quantities, such as the curvature and torsion of certain lines drawn on the middle surface, the most appropriate course to pursue is to endeavor to express the stresses in terms of such geometrical quantities.

There is one class of problems which can often be solved without much difficulty which arise when a plane surface is bent without extension into a developable surface, or when a developable surface is bent into a plane or into some other developable surface such that the lines of curvature on the old surface are lines of curvature on the deformed one. This method can generally be applied when a plane plate is bent into a conical or cylindrical surface; but it could not be applied when a right circular cone is bent into a cone whose lines of curvature are not identical with those of the original one.

The success of this method, in cases where it can be applied, depends upon the fact that the flexural couples G_1, G_2 can be expressed in terms of the changes of c rvature, and also that in the special cases alluded to a sufficient number of the ten stresses are zero to enable the remainder to be determined by means of the general equations of equilibrium.

33. Let OA, OB be two lines of curvature on the middle surface of the undeformed shell; O_1, O_2 the centres of principal curvature; let oa, ob be the curves in which the planes OAO_1, OBO_2 meet any layer of the shell. Let



ρ_1, ρ_2 be the principal radii of curvature at O; let $Oo = \eta'$; also let accented letters denote the strained positions of the various points.

If P denote the normal traction along oa, and R the normal traction along Oo,

$$\begin{aligned} P &= (m+n)\sigma_1 + (m-n)(\sigma_2+\sigma_3) \\ &= 2n\{\sigma_1 + E(\sigma_1+\sigma_2)\} + ER. \end{aligned} \quad (1)$$

Now
$$\sigma_1 = \frac{o'a' - oa}{oa}$$
and
$$\frac{oa}{OA} = 1 + \frac{\eta}{\rho_1}, \quad \frac{o'a'}{O'A'} = 1 + \frac{\eta'}{\rho_1'}, \quad \eta' = \eta(1+\sigma_3).$$

Since we neglect the extension of the middle surface, $O'A' = OA$, whence

$$\sigma_1 = \frac{\eta'/\rho_1' - \eta/\rho_1}{1 + \eta/\rho_1} = \eta\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1}\right) + \frac{\eta\sigma_3}{\rho_1'}$$
$$= \eta\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1}\right) + \frac{\eta}{\rho_1'}\left\{\frac{R}{m+n} - E(\sigma_1 + \sigma_2)\right\}. \tag{2}$$

Similarly,
$$\sigma_2 = \eta\left(\frac{1}{\rho_2'} - \frac{1}{\rho_2}\right) + \frac{\eta}{\rho_2'}\left\{\frac{R}{m+n} - E(\sigma_1 + \sigma_2)\right\}, \tag{3}$$
and the value of G_2 is
$$G_2 = \int_{-h}^{h} P\eta\, d\eta. \tag{4}$$

Now, according to the fundamental hypothesis laid down in §5, it follows that, *provided there is no external pressure*, R must be a quadratic function of h and η, and consequently the retention of R will lead on integration to terms in G_2 involving higher powers of h than h^3; accordingly we may neglect R, and if we substitute the values of σ_1, σ_2 from (2) and (3) in (1), and the resulting value of P in (4) and integrate, we shall obtain

$$G_2 = \tfrac{4}{3}nh^3\left\{(1+E)\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1}\right) + E\left(\frac{1}{\rho_2'} - \frac{1}{\rho_2}\right)\right\}. \tag{5}$$

Similarly we can show that
$$G_1 = -\tfrac{4}{3}nh^3\left\{E\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1}\right) + (1+E)\left(\frac{1}{\rho_2'} - \frac{1}{\rho_2}\right)\right\}. \tag{6}$$

We notice that the above values of G_1 and G_2 agree with those which we have already obtained for plane plates and cylindrical and spherical shells when the bending is small.

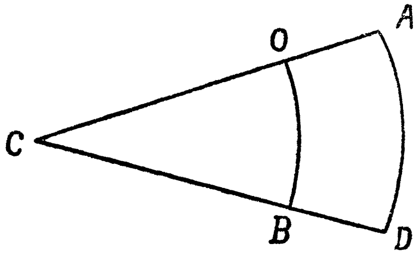
When a plane plate is bent into a developable surface $\rho_1 = \rho_2 = \infty$; also one of the quantities ρ_1' or ρ_2' (say ρ_2') is infinite; whence (5) and (6) become

$$\left.\begin{aligned} G_2 &= \tfrac{4}{3}nh^3(1+E)/\rho_1', \\ G_1 &= -\tfrac{4}{3}nh^3E/\rho_1'. \end{aligned}\right\} \tag{7}$$

Since the extension of the middle surface is neglected, equations (5) and (6) would not apply to the case of a plane plate which is deformed into a surface such as a portion of a sphere.

34. As an example of the preceding method, we shall consider the case of a plane plate of thickness $2h$ which is bounded by two radii CA, CD and two arcs OB, AD of concentric circles; and we shall inquire whether it is possible to bend this plate into a portion of a right circular cone in which OA, BD are generators and OB, AD are circular sections.

We shall assume for trial that the bending may be effected by means of tensions, normal shearing stresses and flexural couples applied to the edges; so that the tangential shearing stresses and the torsional couples are zero.



Let α be the semi-vertical angle of the cone, r the distance of any point on $OADB$ from C. Then

$$\rho_1' = \rho_2' = \infty\,; \quad \rho_2 = \infty\,, \quad \rho_2' = r\tan\alpha\,;$$

whence

$$G_2 = \tfrac{4}{3}nh^3Er^{-1}\cot\alpha,$$

$$G_1 = -\tfrac{4}{3}nh^3(1+E)\,r^{-1}\cot\alpha.$$

The equations of equilibrium are

$$\left.\begin{aligned}
&\frac{d}{dr}(T_1 r) - T_2 = 0\,,\\
&\frac{dT_2}{d\phi} + N_1\cos\alpha = 0\,,\\
&\frac{d}{dr}(N_2 r)\sin\alpha + \frac{dN_1}{d\phi} - T_2\cos\alpha = 0\,,\\
&\frac{dG_1}{d\phi} + N_1 r\sin\alpha = 0\,,\\
&\frac{d}{dr}(G_2 r) - N_2 r + G_1 = 0\,.
\end{aligned}\right\} \tag{8}$$

Let $\qquad \tfrac{4}{3}nh^3(1+E)\cot\alpha = k,$

then $\qquad G_1 = -k/r\,;$

whence $\qquad N_2 = -k/r^2, \quad N_1 = 0\,,$

$$T_2 = k\tan\alpha/r^2,$$

and therefore

$$T_1 = \frac{A}{r} - \frac{k\tan\alpha}{r^2}\,,$$

where A is the constant of integration.

From these equations we see that all the stresses are determinate except T_1'. If $CO = a$, $CA = b$, and T_a, T_b denote the tensions along OB, AD, we have

$$T_a = \frac{A}{a} - \frac{k \tan \alpha}{a^2},$$

$$T_b = \frac{A}{b} - \frac{k \tan \alpha}{b^2},$$

from which it appears that either T_a or T_b may if we please be made zero, provided the other is properly determined.

The above results would also apply to a belt of a complete cone bounded by two circular sections.

FLEDBOROUGH HALL, HOLYPORT, BERKS, ENGLAND.

[CORRECTIONS BY A. CHESSIN.]

Page 165.

Instead of: for every whole negative number a_e there should be another whole negative number $a_{e'}$, and that at the same time $A_e x^{a_{e'}} + A_{e'} x^{a_e} = 0$

read:

$a_{e_1}, a_{e_2}, \ldots, a_{e_\rho}$ being the negative numbers, the expression

$$A_{e_1} x^{-a_{e_1}} + A_{e_2} x^{-a_{e_2}} + \ldots + A_{e_\rho} x^{-a_{e_\rho}}$$

vanishes identically.

Page 182.

Instead of: for every such α_e there should be another whole negative number $\alpha_{e'}$, and that at the same time

$$\frac{A_e x^{-a_e}}{(-a\alpha_e + b)!} + \frac{A_{e'} x^{-a_{e'}}}{(-a\alpha_{e'} + b)!} = 0$$

read:

$\alpha_{e_1}, \alpha_{e_2}, \ldots, \alpha_{e_\rho}$ being the negative numbers, the expression

$$\frac{A_{e_1} x^{-a_{e_1}}}{(-a\alpha_{e_1} + b)!} + \frac{A_{e_2} x^{-a_{e_2}}}{(-a\alpha_{e_2} + b)!} + \ldots + \frac{A_{e_\rho} x^{-a_{e_\rho}}}{(-a\alpha_{e_\rho} + b)!}$$

vanishes identically.